Composition: Part 2 Symmetry Operations

Transformations

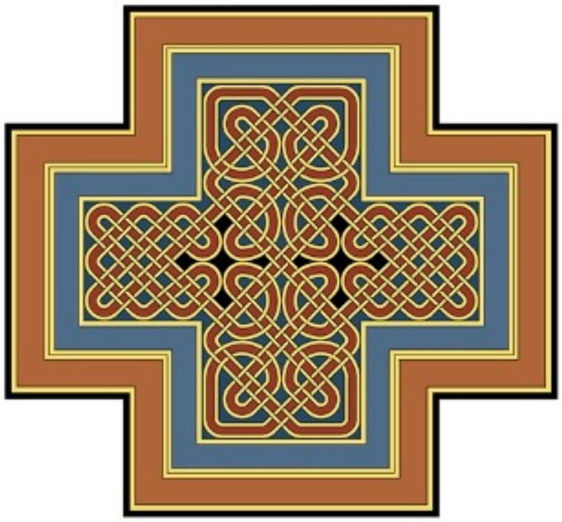

Figure 1. Recreated detail from the Book of Kells folio 291V. Image by Meg Yamamoto. Used with permission.

The image above is a recreation of an interlace from folio 291v from the book of Kells. Although the design looks complex, the interlace can be broken down to reveal a simple pattern that consists of circles and lines. Simply put, the entire interlace can be built by transforming a circle and a line by means of translation, rotation, reflection, and dilation.

First, isolate the interlacing strands:

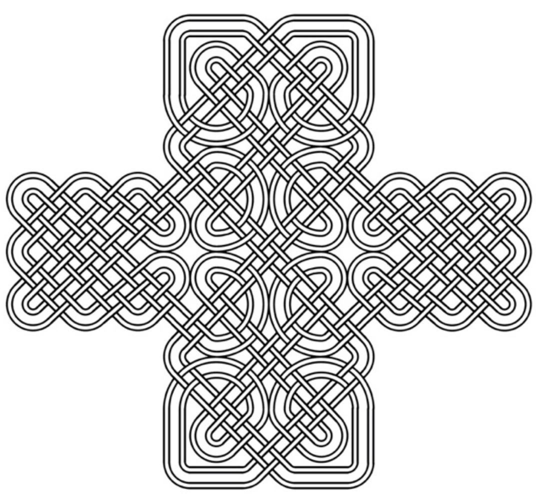

Figure 2. Interlace from the Book of Kells folio 291V in isolation. Image by Meg Yamamoto. Used with permission.

(Note that “corners” are removed from the initial image to emphasize curves.)
Complete each curve in the interlace as a full circle to identify where circles are. Straight lines are removed in the image below to emphasize where concentric circles are located:

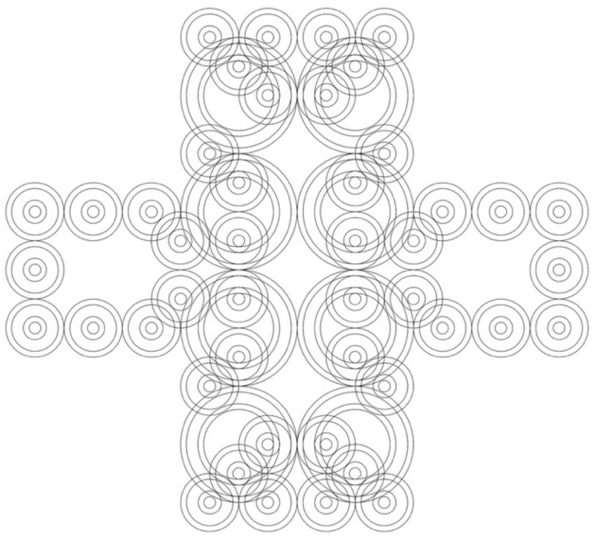

Figure 3. Interlace from the Book of Kells folio 291V. Image by Meg Yamamoto. Used with permission.

1/2, ¼, and 1/8 portions of concentric circles are connected to other concentric circle portions by means of straight lines.

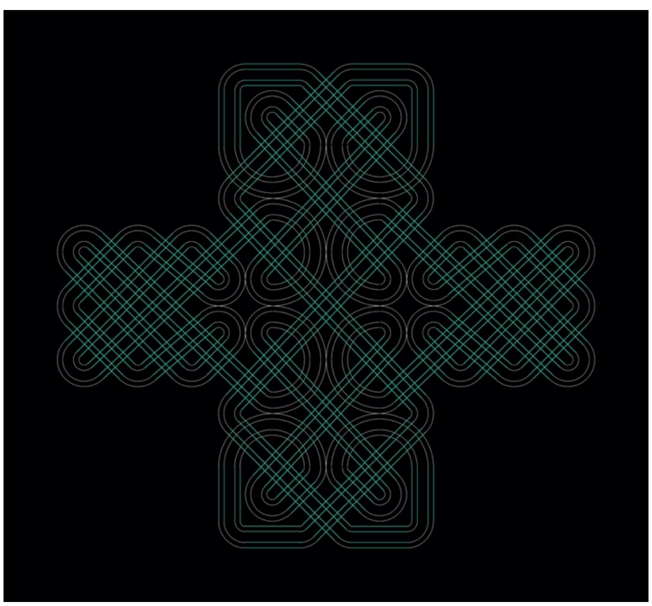

Figure 4. Interlace from the Book of Kells folio 291V. Image by Meg Yamamoto. Used with permission.

In geometry, a transformation is the movement of objects in a coordinate plane.

a **preimage** is transformed in some way to produce an **image**. There are two different kinds of transformations:

1. **Rigid transformation:** does not change the shape or size of a preimage.
2. **Non-rigid transformation**: changes the size but not the shape of a preimage.

Within the rigid and non-rigid transformations, there are four basic types of transformations. Three of them fall in the rigid transformation category; Note that dilation the only non-rigid transformation of the four. For example, when a line is translated as shown in *Figure 1*, the distance between the two points A and B remains the same as the distance between the two points A' and B'. This is a **rigid transformation**.

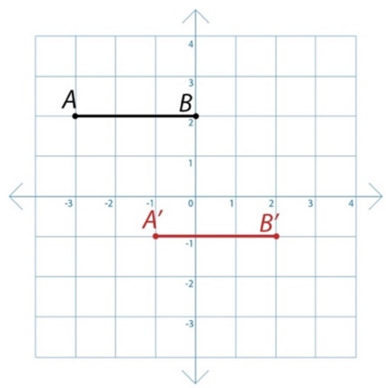

Figure 5. Image by Meg Yamamoto. Used with permission.

1. **Rotation**: an image about a fixed point is rotated (no change to its size or shape)
2. **Translation**: an image is slid from one place to another (no change to its size, shape or orientation)
3. **Dilation**: an image is expanded or contracted (no change to its shape or orientation)
4. **Reflection**: an image is flipped across a line (no change to its size or shape)

An equation is used to transform a point, line, shape, image, or function. Addition or subtraction will cause translations to occur, while multiplication or division will cause dilation.
The Cartesian coordinate system is used in the examples provided below, though the coordinate system for pixels in a computer window positions the ordered pair (0,0) the *Origin* in a different place. In the image below, the *Origin* in the graph on the left (the Cartesian coordinate grid) is in the middle. On the right, the top left pixel of a computer screen is the *Origin*.

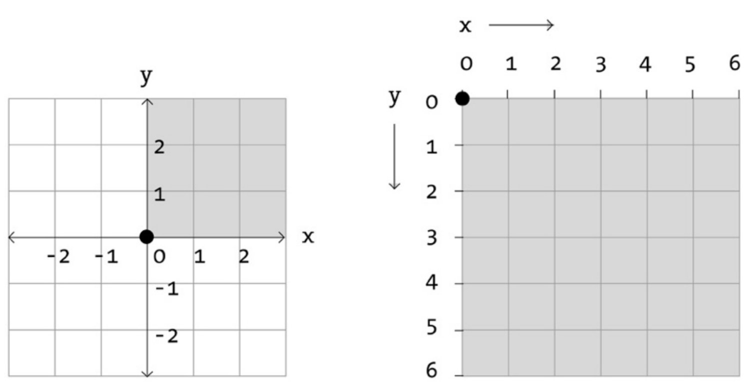

Figure 6. Shiffman, Daniel. *Coordinate System and Shapes*. 2008.

The conceptual use of grids is found in art anywhere where tessellation and patterns are used. Complex use of grids for spacing and creating designs is evident in Insular medieval illuminated manuscripts, such as the Book of Durrow (see *Figure 7* below).

Figure 7. Folio 192V, the Book of Durrow.

Translation

To translate a point, a line, or a shape means to slide it. A *translation* is a transformation which slides each point of a figure the same distance in the same direction.

When an image is translated, nothing is changed in the shape except its coordinates; Translation moves things from one place to another. The shape is not rotated (direction) nor flipped (arrangement), and the scale (size) is not increased nor decreased.

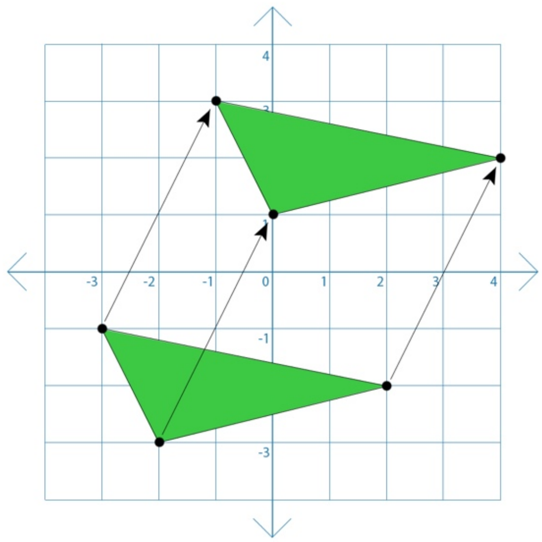

Figure 8. Image by Meg Yamamoto. Used with permission.

If we were to move point A on a Cartesian coordinate grid in *Figure 9* from its location at -2 on the x-axis and 2 on the y-axis, by sliding it 3 units to the right and 2 units down, we translate point A from (-2,2) to A′ (1, -1).

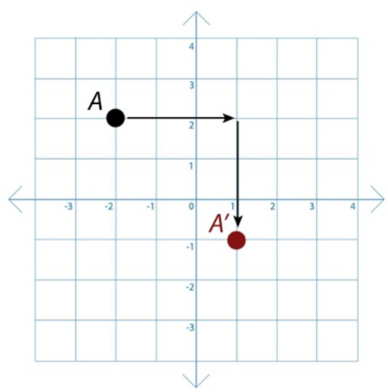

Figure 9. Image by Meg Yamamoto. Used with permission.

In a transformation, the figure on which we are performing the translation is called the *Pre-image figure*. The result of the translation is called the *image figure*. In *Figure 5*, A is the *pre-image figure* and A' is the *image figure*. The same letter is used to identify the pre-image and image figures (to avoid confusion when more points are added), and the image figure has a "prime" suffix added (the image figure becomes A prime). The location of a point is described as an *ordered pair*, its location on the x-axis, a comma, and its location on the y-axis by the format (x,y).

On the cartesian grid, translating a point from (0,0) to (0,2) indicates a slide two units upwards; from (0,0) to (0,-2) indicates a slide two units downwards; (0,0) to (2,0) indicates a slide two units to the right, and (0,0) to (-2,0) indicates a slide two units to the left.
If a point were translated by 3 units to the right and 3 units down will give us A', which can be written as $T_{(3,-3)}$.

$y = f(x) + b$
describes a vertical translation up or down by b units.
When $b > 0$, the image is translated up b units.
When $b < 0$, the image is translated down by b units.

$y = f(x – p)$
describes a horizontal translation right or left by p units.
When $p > 0$, the image is translated right by p units.
When $p < 0$, the image is translated left by p units.

$y – k = f(x – h)$
or
$y = f(x – h) + k$

Activity

In the value tables given below, table a) shows values for graphing a parabola $y = x^2$.
How do the values of tables b) and c) compare to a)?

a)

x	y
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9

b)

x	y
−3	12
−2	7
−1	4
0	3
1	4
2	7
3	12

c)

x	y
−3	6
−2	1
−1	−2
0	−3
1	−1
2	1
3	6

Figure 10. Image by Meg Yamamoto. Used with permission.

In table b), all y values are 3 values *higher* than in $y = x^2$. Therefore, the translation can be described as $y = x^2 + 3$ or $y - 3 = x^2$. (a vertical translation of 3 units up).

In table c), all the y values are 3 values *lower* than the values in $y = x^2$. Therefore, the translation is described as $y = x^2 - 3$ or $y + 3 = x^2$. (a vertical translation of 3 units down).

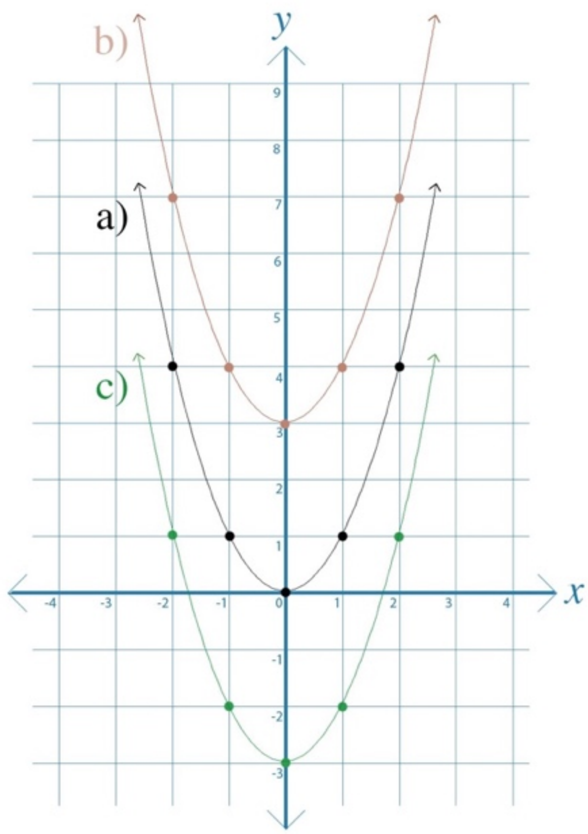

Figure 11. Image by Meg Yamamoto. Used with permission.

In the tables below, we see horizontal transformations of the function $y = x^2$:

a)

x	y
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9

b)

x	y
0	9
1	4
2	1
3	0
4	1
5	4
6	9

c)

x	y
−6	9
−5	4
−4	1
−3	0
−2	1
−1	4
0	9

Figure 12. Image by Meg Yamamoto. Used with permission.

In table b), x has been replaced with x — 3: the *p* value is 3, represented as $y = (x - 3)^2$ (which will graph as a horizontal translation of three units right).
In table c), x has been replaced with x + 3; the *p* value is -3; we see a horizontal translation of three units left: $y = (x + 3)^2$.

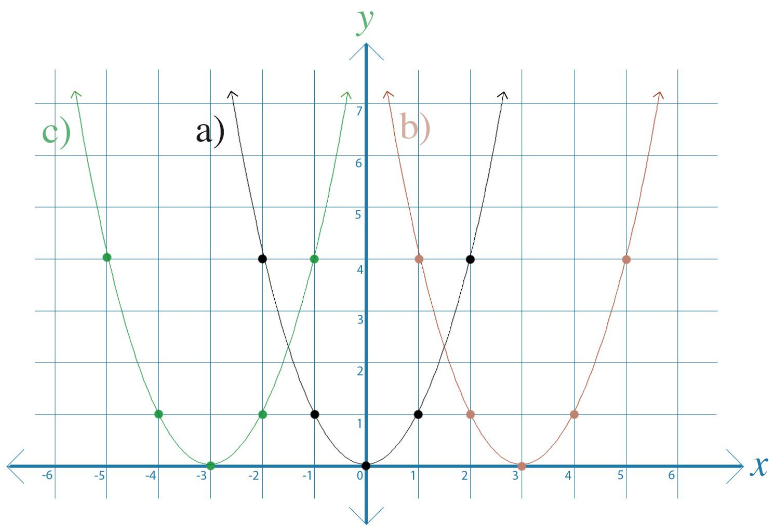

Figure 13. Image by Meg Yamamoto. Used with permission.

Comparing functions:

In the example below, f(x) is shown in red and g(x) is shown in purple.
If we compare the minimum point, we can see that the minimum point is horizontally shifted by -4 (four units to the left) and -7 vertical shift (seven units down)

g(x) = f(x – horizontal shift) + vertical shift

so g(x) = f(x – -4) + -7 or g(x) = f(x+4) -7

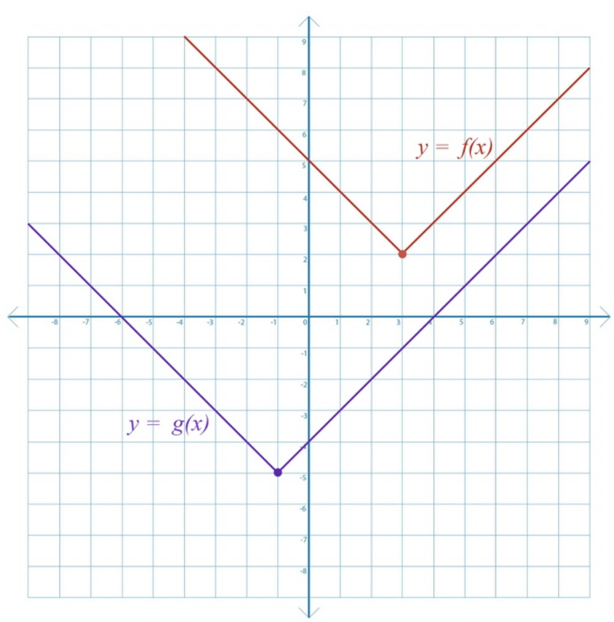

Figure 14. Image by Meg Yamamoto. Used with permission.

Consider how translations are used to create the arrangement of forms in the designs below:

Figure 15. Image by Meg Yamamoto. Used with permission.

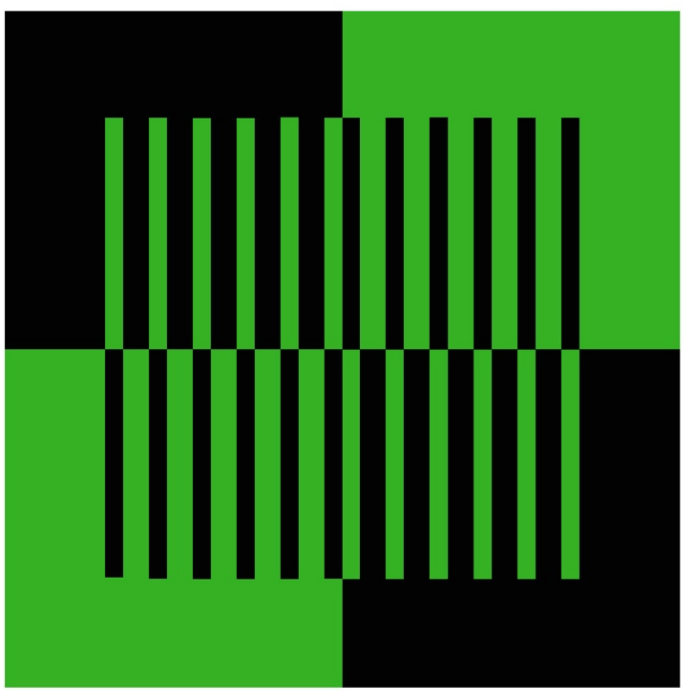

Figure 16. Image by Meg Yamamoto. Used with permission.

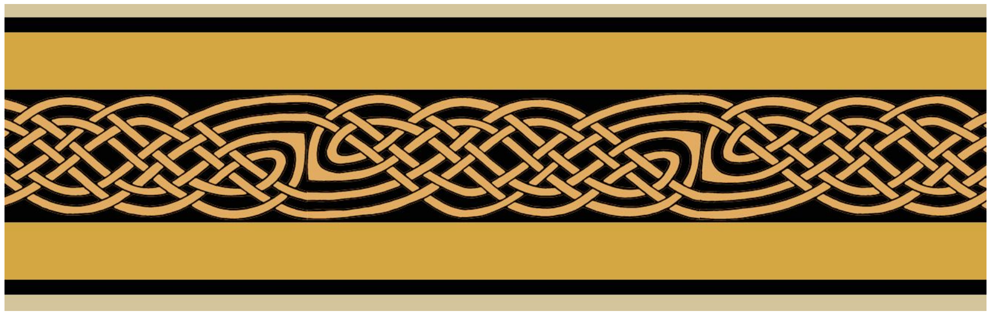

Figure 17. Image by Meg Yamamoto. Used with permission.

Consider the recreated cannon table detail above, from the Book of Kells folio 5r. This interlace is segmented by pattern and one segment is isolated and shown below:

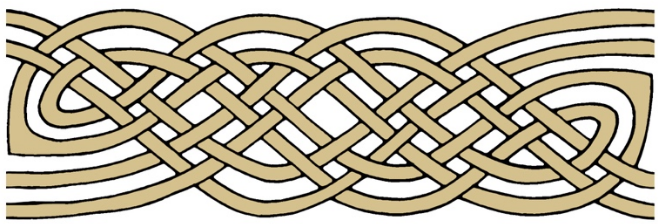

Figure 18. Image by Meg Yamamoto. Used with permission.

This segment can actually be simplified further by other means of transformations but for the purpose of examining translations, we'll leave it in this form. Let's say that this segment is one unit high and three units wide.

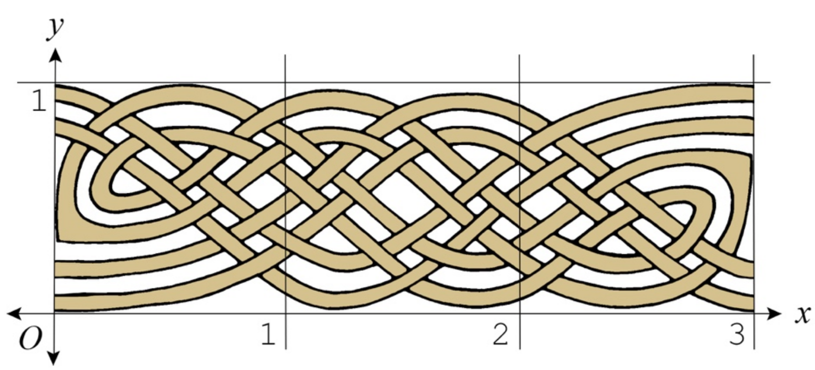

Figure 19. Image by Meg Yamamoto. Used with permission.

This form can be horizontally translated by three units, and then by six units, and then nine units...

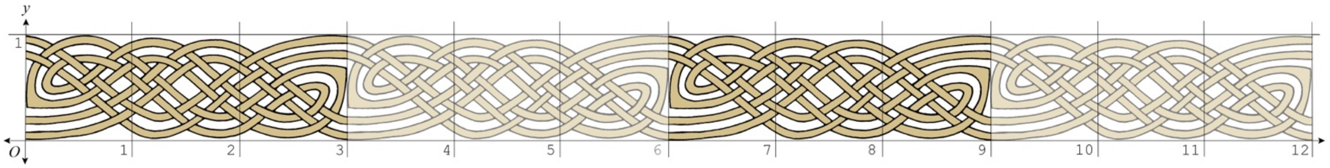

Figure 20. Image by Meg Yamamoto. Used with permission.

to create a more complex design:

Figure 21. Image by Meg Yamamoto. Used with permission.

Printmaking

Figure 22. Cueva de las Manos, Perito Moreno, Argentina.

The concept of translation as a means of producing an image goes all the way back to cave paintings. Consider how the image above was created – a hand is used as a stencil, and paint is applied around the hand. The hand is then translated vertically and or horizontally to a different place on the stone surface, and paint is applied again (assuming the same hand is used).

The discipline of printmaking focuses heavily around the concept of translation: An image is created so that it can be placed onto a different place: on a different sheet of paper, on a sheet of fabric...

This technique of reproducing an image multiple times by means of impression was utilized by the Sumerians (c. 3000 BCE), who engraved designs and cuneiform inscriptions on cylinder seals (usually made of stone), which, when rolled over soft clay tablets, left relief impressions (www.britannica.com/art/printmaking/History-of-printmaking (Links to an external site.)).

Figure 23. Sumerian cylinder seal.

Cylinder seals were impression stamps, often quite intricate in design, used throughout Mesopotamia. They were known as *kishib* in **Sumerian** (Links to an external site.) and *kunukku* in Akkadian and were used by everyone, from royals to slaves, in the transaction of business and sending correspondence. They originated in the Late **Neolithic Period** (Links to an external site.) c. 7600-6000 BCE in the region known today as **Syria** (Links to an external site.) (though, according to other claims, they originated in **Sumer** (Links to an external site.), modern Iraq, sometime later) and were made from semiprecious stone (such as marble, obsidian, amethyst, lapis lazuli) or **metal** (Links to an external site.) (**gold** (Links to an external site.) or **silver** (Links to an external site.)). These seals were worn by their owners on strings of leather or other material around the neck or wrist or pinned to a garment. Their purpose was to serve as a personal signature on a document or package to guarantee authenticity or legitimize a business deal as one signs a letter or form in the present day. The seal was rolled onto the moist clay of the document as an official, binding signature (www.ancient.eu/article/846/cylinder-seals-in-ancient-mesopotamia---their-hist/ (Links to an external site.)).

{\displaystyle r=a+b\theta }Printmaking is an artistic process based on the principle of transferring images from a matrix onto another surface, most often paper or fabric. Traditional printmaking techniques include woodcut, etching, engraving, and lithography, while modern artists have expanded available techniques to include screenprinting... One of the great benefits of printmaking (save for monotype) is that multiple impressions of the same design can be printed from a single matrix. (www.metmuseum.org/about-the-met/curatorial-departments/drawings-and-prints/materials-and-techniques/printmaking#:~:text=Printmaking%20is%20an%20artistic%20process,available%20techniques%20to%20include%20screenprinting. (Links to an external site.)).

In these examples of image-making, nothing in the form of the image is changed – only the location of the image is new. This technology of making woodblock prints, etchings, engravings, among others allows an image to be translated many times.

Figure 24. Albrecht Dürer . The Four Horsemen of the Apocalypse, 1498.

Figure 25. Albrecht Dürer - Melancholia I, 1514. Engraving.

Figure 26. MC Escher *Waterfall*, 1961. Lithograph.

In many cases, the actual transferred image that is printed is a reflected image of what is on the plate, woodblock, stone... However, the actual printed image is translated as it is printed multiple times.

Spirals and Symmetry Operations

Spirals are observed in the form of “embryos, horns, whirlpools, hurricanes and galaxies, the path that energy takes when left alone, the path of unfettered yet balanced growth. Spiral motifs appear worldwide in the symbolism of religion, art, dreams, folktales and mythology. mathematically, a spiral is simply a line that grows continuously toward or away from its own centre. But its symbolic power is in the evocation of an archetypal path of growth, transformation and psychological or spiritual journey. Based on the direction of its spin, whether expanding outward and larger, or tightening inward and smaller, a spiral is a cosmic symbol that may represent one or the other of several dualities: growth or decay, ascent or descent, evolution or involution, waxing or waning, accumulation or dissolution, increasing or decreasing, exxpanding or contracting, offering or recieving, revealing or hiding. The double spiral combines both opposites in one glyph.”
(pg. 718)
Carlson, K et al. *The Book of Symbols: Reflections on Archetypal Images*, Taschen, 2010

Archimedean Spiral – commonly depicted in art due to varying reasons such as the nature of pottery, observations in the natural world (mathworld.wolfram.com/ArchimedeanSpiral.html (Links to an external site.)).

Concentric circles and Spirals are symbols used in art for thousands of years, and almost universally across cultures. Celtic art is known for its spirals, including the single strand spiral, double-strand, and triple strand.

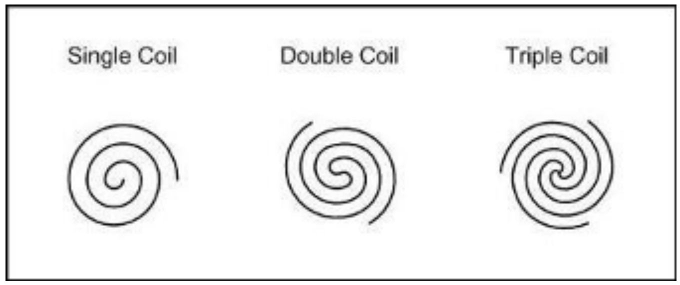

Figure 27. Single, double, and triple coil spirals.

Figure 28. Single, Double, and Triple Spirals.

A set of concentric circles are shown below. The innermost circle is two units wide and each successive circle increases in diameter by two units. The left half of the concentric circles are translated 1 unit down to form two spiral lines, one starting at (0,0) and ending at (0,-6) shown in green, and the other beginning at (0,-1) and (0,5) shown in red.

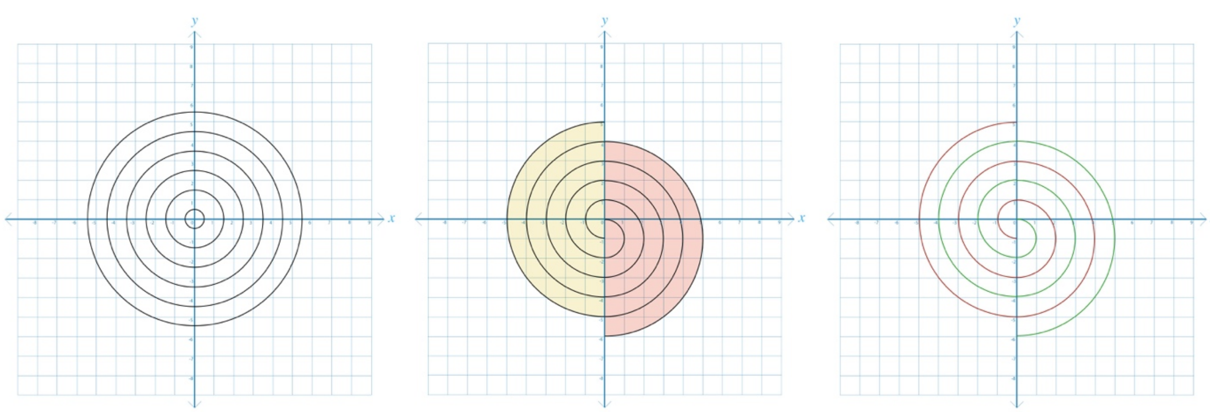

Figure 29. Image by Meg Yamamoto. Used with permission.

Similarly, a translation of 1 unit vertically or horizontally to half of a series of regularly-spaced concentric circles starting with the innermost circle being one unit in diameter will produce a double coil strand.

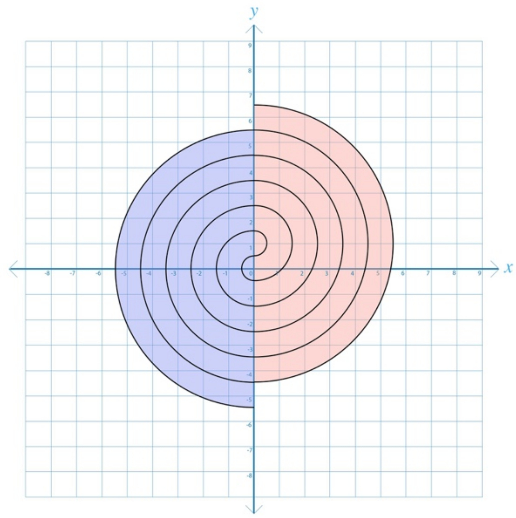

Figure 30. Image by Meg Yamamoto. Used with permission.

Three overlapping sets of concentric circles illustrate how a triple coil knot emerges.

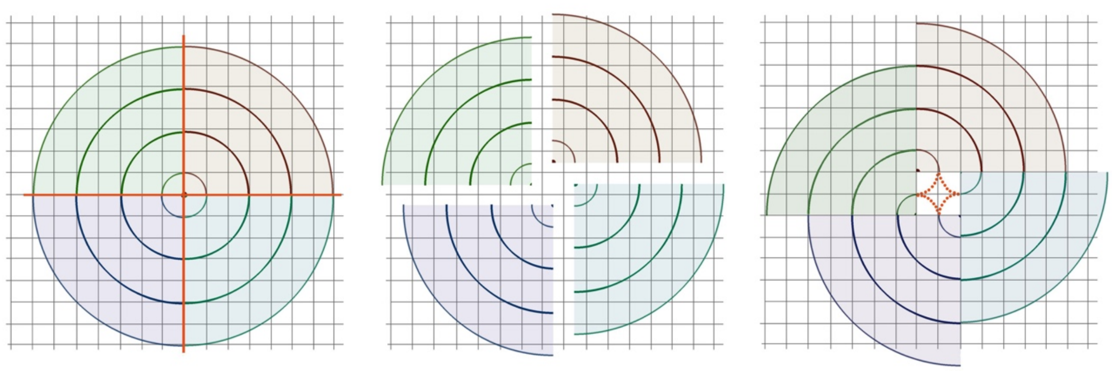

Figure 31. Image by Meg Yamamoto. Used with permission.

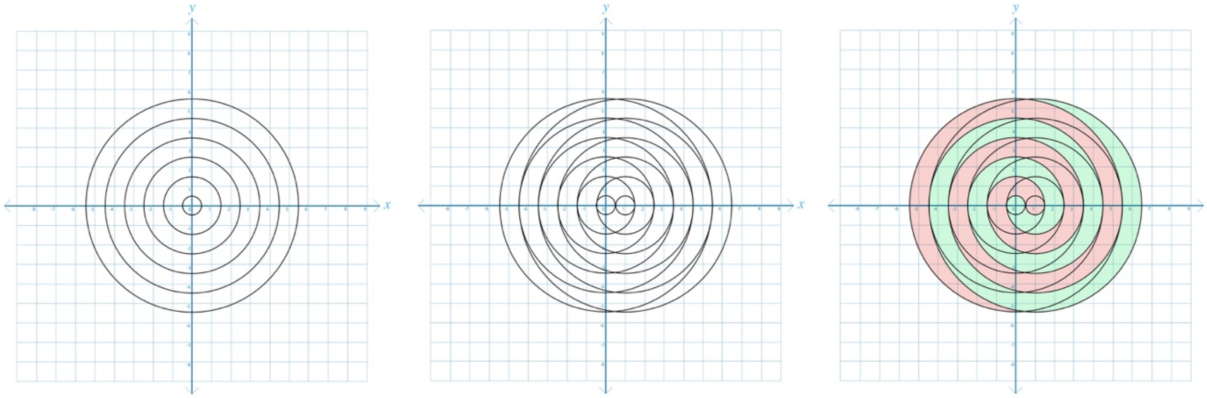

Figure 32. Image by Meg Yamamoto. Used with permission.

Rather than imagining one set of concentric circles being cut in half, it is also possible to imagine the entire set of concentric circles being translated to overlap the pre-image figure. The two spiralling coils that emerge travel between the boundaries of both sets of concentric circles. Consider how these examples of spiral construction (overlapping multiple concentric circles or dividing a single set of concentric circles might influence the symbolic meaning behind the spiral.

The Archimedean spiral is the trajectory of a point moving uniformly on a straight line of a plane, this line turning itself uniformly around one of its points (carried out for example by the groove of a good old vinyl disk); here, O is the center of rotation, r = 0 for q = 0.

The **Archimedean (arithmetic) spiral** is named after 3rd cen BC Greek (Links to an external site.) mathematician (Links to an external site.) Archimedes (Links to an external site.). It is the locus (Links to an external site.) of points corresponding to the locations over time of a point moving away from a fixed point with a constant speed along a line that rotates with constant angular velocity (Links to an external site.). Equivalently, in polar coordinates (Links to an external site.) (r, θ) it can be described by the equation $r = a + b\,\theta$ with real numbers (Links to an external site.) a and b. Changing the parameter a moves the centerpoint of the spiral outward from the origin (positive a toward $\theta=0$ {\displaystyle \theta =0} and negative a toward $\theta= pi$ {\displaystyle \theta =\pi }), while b controls the distance between loops.

From the above equation, it can thus be stated: the position of particle from the point of start is proportional to the angle θ {\displaystyle \theta } as time elapses. (en.wikipedia.org/wiki/Archimedean_spiral).

Some other spirals

Fibonacci spiral

In mathematics, the Fibonacci numbers form a sequence defined by the following recurrence relation. The Fibonacci numbers first appeared, under the name *matrameru* (mountain of cadence), in the work of the Sanskrit grammarian Pingala (Chandah-shastra, the Art of Prosody, 450 or 200 BC). The Fibonacci sequence is named after Italian mathematician Leonardo of Pisa, known as Fibonacci. His 1202 book Liber Abaci introduced the sequence to Western European mathematics (www.crystalinks.com/fibonacci.html).

A Fibonacci spiral starts with a rectangle partitioned into 2 squares. In each step, a square the length of the rectangle's longest side is added to the rectangle. consider how symmetry operations are combined to construct the Fibonacci spiral.

Golden spiral: a **golden spiral** is a logarithmic spiral (Links to an external site.) whose growth factor is φ (Links to an external site.), the golden ratio (Links to an external site.).[1] (Links to an external site.) That is, a golden spiral gets wider (or further from its origin) by a factor of φ for every quarter turn it makes.

Fermat's spiral

A **Fermat's spiral** or **parabolic spiral** is a plane curve (Links to an external site.) named after Pierre de Fermat (Links to an external site.).[1] (Links to an external site.) It describes a parabola (Links to an external site.) with a horizontal axis. (mathworld.wolfram.com/FermatsSpiral.html (Links to an external site.)).

Rotation

In rotation, the *direction* of the image changes. The image turns about to a fixed point. This fixed point is called the *centre of rotation*. Rotations may occur clockwise or counterclockwise.
On a coordinate grid, the degree movement follows a counterclockwise direction.

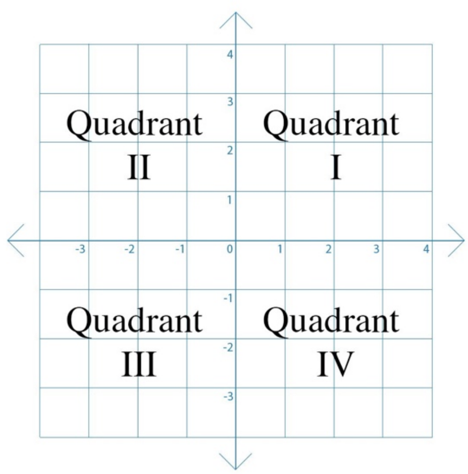

Figure 33. Image by Meg Yamamoto. Used with permission.

Generally, the centre of rotation on a coordinate plane is the Origin (0,0) unless otherwise specified.
A positive rotation moves counterclockwise: for example, if the pre-image point A is located at (3,0) on the coordinate grid, a 90-degree rotation about the origin would place the image point at (0,3).

Activity

1. In the image below a paving stone is represented on the left. Two ¼ circles in opposing corners decorate the tile. On the right is one potential pattern that can be created using these tiles.

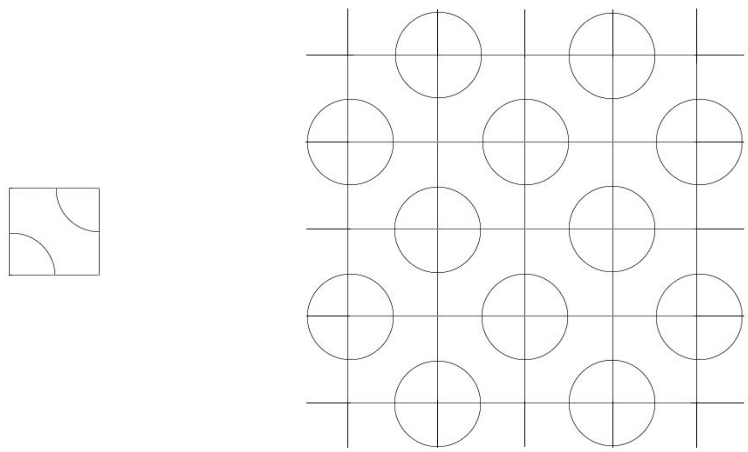

Figure 34. Image by Meg Yamamoto. Used with permission.

What other patterns and designs can you create from these tiles using rotation? View some other patterns here: www.mi.sanu.ac.rs/vismath/asakura/p116.htm (Links to an external site.).

2. **Examine** how rotation is used to create the interlaced design below:

Figure 35. Reproduction of detail from folio 114r, *Book of Kells*. Image by Meg Yamamoto. Used with permission.

Dilation (scale)

Expansion, “Steckung" (stretching). This operation strengthens a dynamic feeling in visual design.
A dilation is a transformation that transforms a pre-image to an image of the same shape, but of a different size. A dilation can enlarge or shrink an image, but all proportions remain the same.
It requires a center point and a scale factor, k. The value of k determines whether the dilation is an enlargement or a reduction.
“A dilation is a transformation that produces an image that is the same shape as the original but is a different size. (The image is similar to the original object). Dilation is a transformation in which each point of an object is moved along a straight line. The straight line is drawn from a fixed point called the center of dilation. The distance the points move depends on the scale factor. The center of dilation is the only invariant point.”
(https://www.onlinemathlearning.com/dilation-transformation.html (Links to an external site.))

- A description of a dilation includes the scale factor (or ratio) and the center of the dilation.
- The center of dilation is a fixed point in the plane.
- If the scale factor is greater than 1, the image is an enlargement.
- If the scale factor is between 0 and 1, the image is a reduction.

(https://mathbitsnotebook.com/Algebra1/FunctionGraphs/FNGTransformationDilation.html (Links to an external site.))

If the scale factor is 1, the figure and the image are congruent.

To dilate a figure with respect to the origin, multiply the coordinates of each vertex by the scale factor k.
(x, y) to (kx, ky)
When $|k| > 1$, the dilation is an enlargement.
When $|k| < 1$, the dilation is a reduction.

The absolute value of the scale factor determines the size of the new image as compared to the size of the original image. When k is positive the new image and the original image are on the same side of the center. When k is negative they are on opposite sides of the center. The center of a dilation is always its own image.

Dilations preserve angle measure, betweenness of points and collinearity. It does not preserve distance.

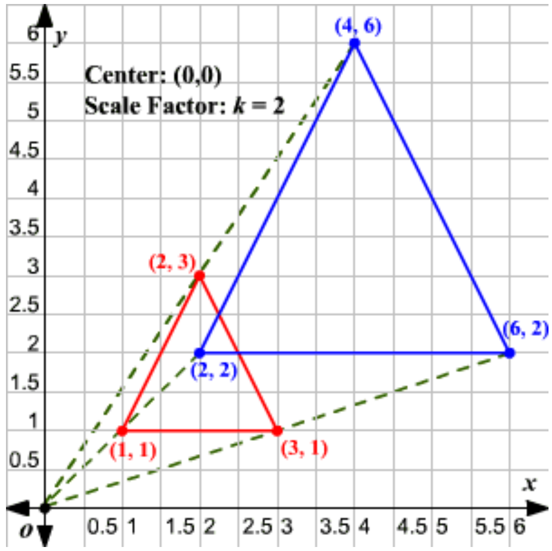

Figure 36. Dilation.

Reflection

1. y = -f(x) is a reflection on the x-axis. (x , y) -> (x , - y)

so when the negative is in front of the entire function or outside of the function, there is a reflection on the x-axis.

y = f(x) -> y = -f(x)

1. y = f(-x) is a reflection on the y-axis. (x , y) -> (- x , y)

if a negative is in front of the x or inside the function, there is a reflection on the y-axis.

y = f(x) -> y = f(-x)

1. y = f^{-1}(x) or x = f(y) is a reflection on the line y = x. (x , y) -> (y , x)

The function is reflected on the line y = x when the *inverse* of a function is graphed.

y = f(x) -> y = f^{-1}(x)

Reflection in the x-axis: If the graph of y=f(x) is reflected in the line y=0, then it is the graph of –f(x) : every output y value and making it negative. Y=-f(x).
Replacing y with -y describes a reflection in the x-axis. -y=f(x) or y=-f(x) describes a reflection in the x axis.

Reflection in the y-axis: If the graph of y=f(x) is reflected in the line x=0, than it is the graph of y=f(-x) : every output x is made negative /x is being replaced with -x. y= f(-x)
Replacing x with -x describes a reflection in the y-axis. Y=f(-x) describes a reflection in the y-axis.

Reflection in the line y=x: If the graph of y=f(x) is reflected in the line y=x, then it is the graph of x=f(y) or $y=f^{-1}(x)$.
Interchanging x and y describes a reflection in the line y=x. x=f(y) or $y=f^{-1}(x)$ describes a reflection in the line y=x.

To find an inverse, switch x and y and solve for y:

$y = x^2$

$x = y^2$

$\sqrt{x} = \sqrt{y^2}$

$y = \pm\sqrt{x}$

Below is the graph y = f(x).

a) Show y = -f(x). (y is replaced with -y)

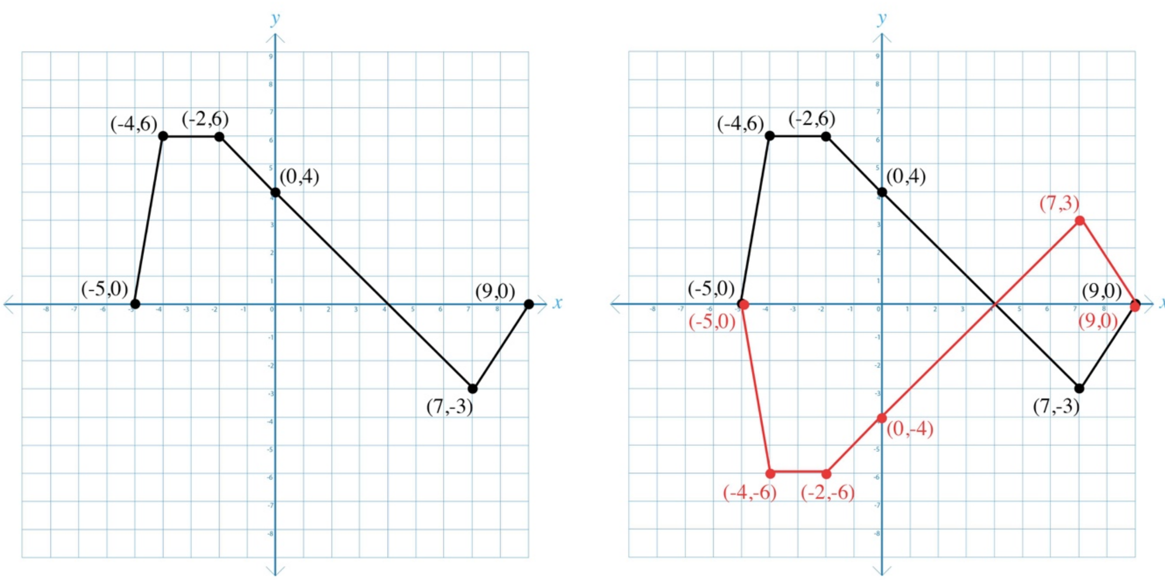

Figure 37. Image by Meg Yamamoto. Used with permission.

b) y = f(-x)

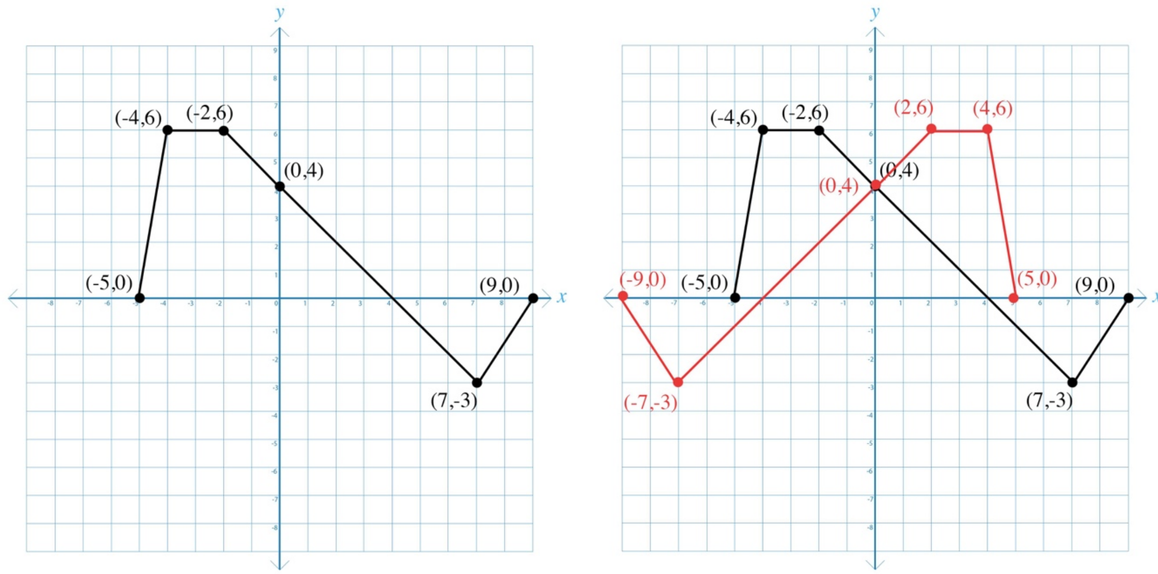

Figure 38. Image by Meg Yamamoto. Used with permission.

c) x = f(y)

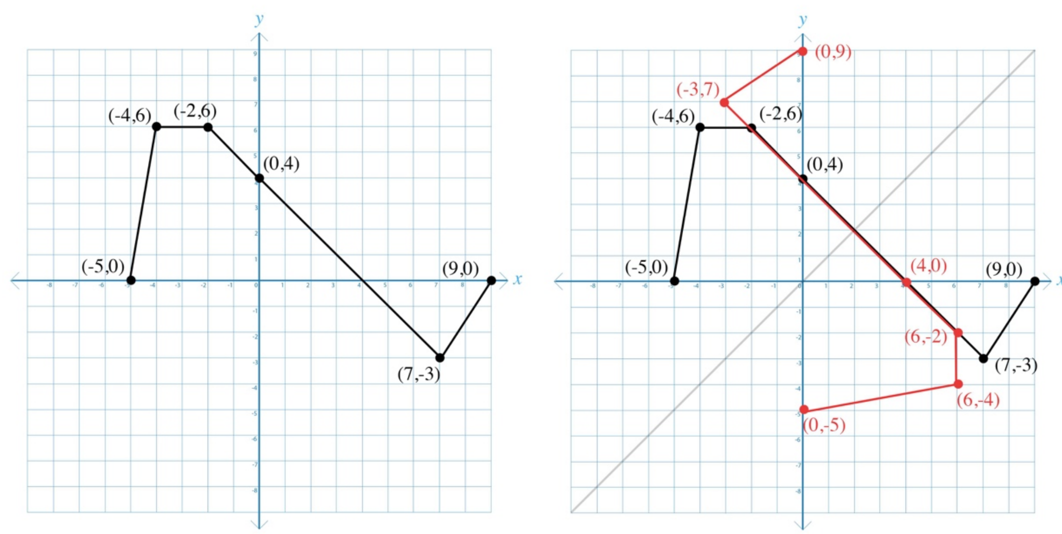

Figure 39. Image by Meg Yamamoto. Used with permission.

Activity

1) **Describe** how reflection is used in the image below:

Figure 40. Lindisfarne Gospel Carpet Page.

2) **Read** elib.mi.sanu.ac.rs/files/journals/vm/56/vm_2.pdf (Links to an external site.)
Marinković, Zorica and Biljana Stojičić. “Axial reflection and plane mirror reflection in analytic geometry.” n.d. elib.mi.sanu.ac.rs/files/journals/vm/56/vm_2.pdf.

Other Transformations

Advancing Reflection: This operation, sometimes also called *Slipping Reflection* or *Glide Reflection*, simultaneously uses translation and reflection to create a pattern. The two designs below are examples of advancing reflection.

Figure 41. Shiratori, Mitsuko. “Figure 291: Symmetry of "advancing reflection”. *Fundamental Problems of Creating in Two-Dimensional Space* by Naomi Asakura. 1992.

Activity

1) **Consider** the design below to identify the operations are used to produce the image:

Figure 42. Katsui, Mitsuo. “Figure 295: “A pattern which includes operations of "expansion", "rotation", and "movement", "Samayoi."” *Fundamental Problems of Creating in Two-Dimensional Space* by Naomi Asakura. 1992.

2) **Read** Nelson, Anna, Holli Newman and Molly Shiply. "17 Plane Symmetry groups." caicedoteaching.files.wordpress.com/2012/05/nelson-newman-shipley.pdf

2) **Read** www.sciencedirect.com/science/article/pii/0898122186904359 (Links to an external site.) (Links to an external site.)

Makovicky, Emil. "Symmetrology of art: coloured and generalized symmetries." *Computers & Mathematics with Applications*, Vol.12, No 3–4 Part 2,1986, pp. 949-980.

Figures:

Figure 1. Recreated detail from the Book of Kells folio 291V. Image by Meg Yamamoto. 2020. Used with permission.

Figure 2. Interlace in isolation. Image by Meg Yamamoto. 2020. Used with permission.

Figure 3. Image by Meg Yamamoto. 2020 .Used with permission.

Figure 4. Image by Meg Yamamoto. 2020. Used with permission.

Figure 5. Image by Meg Yamamoto. 2020. Used with permission.

Figure 6. Shiffman, Daniel. Coordinate System and Shapes. P5.js. 2008. p5js.org/learn/coordinate-system-and-shapes.html. Accessed 1 July 2020.

Figure 7. “Folio 192v of the Book of Durrow.” *Wikipedia*. 29 March 2020. en.wikipedia.org/wiki/Book_of_Durrow#/media/File:Meister_des_Book_of_Durrow_002.jpg (Li nks to an external site.). Accessed 4 July 2020.

Figure 8. Image by Meg Yamamoto. 2020. Used with permission.

Figure 9. Image by Meg Yamamoto. 2020. Used with permission.

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Figure 19. Image by Meg Yamamoto. 2020. Used with permission.

Figure 20. Image by Meg Yamamoto. 2020. Used with permission.
Figure 21. Image by Meg Yamamoto. 2020. Used with permission.

Figure 22. Cueva de las Manos, Perito Moreno, Argentina, 13,000–9,000 BP.
Photo by Cecowski, Mariano. "File:SantaCruz-CuevaManos-P2210651b.jpg." Wikimedia Commons. 2 June 2006. en.wikipedia.org/wiki/Cueva_de_las_Manos#/media/File:SantaCruz-CuevaManos-P2210651b.jpg. Accessed 1 July 2020.

Figure 23. Photo by McPhee, Nic. "File:Flickr - Nic's events - British Museum with Cory and Mary, 6 Sep 2007 - 185.jpg." Wikimedia Commons. 6 October 2015. en.wikipedia.org/wiki/Cylinder_seal#/media/File:Flickr_-_Nic's_events_-_British_Museum_with_Cory_and_Mary,_6_Sep_2007_-_185.jpg. Accessed 1 July 2020.

Figure 24. Met Museum. "The Four Horsemen, from The Apocalypse, 1498, Albrecht Dürer ." The MET.www.metmuseum.org/art/collection/search/336215 (Links to an external site.). Accessed 1 July 2020.

Figure 25. Albrecht Dürer - Melancholia I, 1514. Engraving.
Photo by DcoetzeeBot. "1514 wood engraving by Albrecht Dürer." Wikimedia Commons. 17 November 2013. en.wikipedia.org/wiki/Melencolia_I#/media/File:Albrecht_D%C3%BCrer_-_Melencolia_I_-_Google_Art_Project_(_AGDdr3EHmNGyA).jpg (Links to an external site.). Accessed 5 July 2020.

Figure 26. MC Escher Waterfall, 1961. lithograph

Figure 27. Lovegodbob. "The Spiral." 7 December 2014. lovegodbob.wordpress.com/tag/double-coil (Links to an external site.)/ (Links to an external site.). Accessed 1 July 2020.

Figure 28. Single, Double, and Triple Spirals. *Tattoo Designs and Symbols*.
n.d. www.vanishingtattoo.com/tattoos_designs_symbols_spirals.htm (Links to an external site.). Accessed 1 July 2020.

Figure 29. Image by Meg Yamamoto. 2020. Used with permission.

Figure 30. Image by Meg Yamamoto. 2020. Used with permission.

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Figure 32. Image by Meg Yamamoto. 2020. Used with permission.

Figure 33. Image by Meg Yamamoto. 2020. Used with permission.

Figure 34. Image by Meg Yamamoto. 2020. Used with permission.

Figure 35. Reproduction of detail from folio 114r, Book of Kells. Image by Meg Yamamoto. 2020. Used with permission

Figure 36. Varsity Tutors. n.d. www.varsitytutors.com/hotmath/hotmath_help/topics/dilation (Links to an external site.). Accessed 5 July 2020.

Figure 37. Image by Meg Yamamoto. 2020. Used with permission.

Figure 38. Image by Meg Yamamoto. 2020. Used with permission.

Figure 39. Image by Meg Yamamoto. 2020. Used with permission.

Figure 40. Lindisfarne Gospel Carpet Page. British Library. www.bl.uk/manuscripts/Viewer.aspx?ref=cotton_ms_nero_d_iv_f002r (Links to an external site.). Accessed 5 July 2020.

Figure 41. Shiratori, Mitsuko. “Figure 291: Symmetry of "advancing reflection”. Page 151. *Fundamental Problems of Creating in Two-Dimensional Space* by Naomi Asakura. 1992. www.mi.sanu.ac.rs/vismath/asakura/p129.htm (Links to an external site.) Accessed 15 July 2020.

Figure 42. Katsui, Mitsuo. “Figure 295: “A pattern which includes operations of "expansion", "rotation", and "movement", "Samayoi."” *Fundamental Problems of Creating in Two-Dimensional Space* by Naomi Asakura. 1992. www.mi.sanu.ac.rs/vismath/asakura/p152.htm (Links to an external site.). Accessed 15 July 2020.